

# HYPERGEOMETRIC IDENTITIES FOR 10 EXTENDED RAMANUJAN-TYPE SERIES

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**ABSTRACT.** We prove, by the WZ-method, some hypergeometric identities which relate ten extended Ramanujan type series to simpler hypergeometric series. The identities we are going to prove are valid for all the values of a parameter  $a$  when they are convergent. Sometimes, even if they do not converge, they are valid if we consider these identities as limits.

## 1. INTRODUCTION

In this paper we are going to prove some hypergeometric identities by using WZ-pairs [10], [11], that is, functions  $F(n, k)$  and  $G(n, k)$  related by

$$G(n, k+1) - G(n, k) = F(n+1, k) - F(n, k).$$

A package, written by D. Zeilberger and called *ekhad* [8], allows one to obtain  $F$  from  $G$  or viceversa. For our purposes we need the following implication:

$$\begin{aligned} G(n, k+1) - G(n, k) &= F(n+1, k) - F(n, k) \\ \implies G(n+a, k+1) - G(n+a, k) &= F(n+a+1, k) - F(n+a, k). \end{aligned}$$

So, if we denote  $F_a(n, k) = F(n+a, k)$  and  $G_a(n, k) = G(n+a, k)$  then obviously  $F_a(n, k)$  and  $G_a(n, k)$  is an WZ-pair for every value of  $a$ . So, we can write:

$$G_a(n, k+1) - G_a(n, k) = F_a(n+1, k) - F_a(n, k)$$

and summing from  $n = 0$  to  $\infty$  we get

$$\sum_{n=0}^{\infty} [G_a(n, k+1) - G_a(n, k)] = \sum_{n=0}^{\infty} [F_a(n+1, k) - F_a(n, k)] = -F_a(0, k)$$

which implies that

$$\begin{aligned} \sum_{n=0}^{\infty} G_a(n, 0) &= \sum_{n=0}^{\infty} G_a(n, 1) + F_a(0, 0) = \sum_{n=0}^{\infty} G_a(n, 2) + F_a(0, 1) + F_a(0, 0) \\ &= \sum_{n=0}^{\infty} G_a(n, 3) + F_a(0, 2) + F_a(0, 1) + F_a(0, 0) \\ &= \sum_{n=0}^{\infty} G_a(n, 4) + \sum_{k=0}^3 F_a(0, k) \end{aligned}$$

and continuing the recursion we arrive to [2]:

$$\sum_{n=0}^{\infty} G_a(n, 0) = \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G_a(n, k) + \sum_{k=0}^{\infty} F_a(0, k),$$

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which is the result we are going to use to get the identities. Finally some easy transformations of  $G_a(n, 0)$  and  $F_a(0, k)$  into rising factorials lead to the desire form of the identities and the proof is complete.

## 2. IDENTITIES FOR SOME EXTENDED RAMANUJAN SERIES

We consider an extension with a variable  $a$  of some Ramanujan series for  $1/\pi$  [3], [4] and [9] and get some identities. Once we find an adequate WZ-pair then the difficulty of the proof consist of getting the function

$$S(a) = \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G_a(n, k)$$

and very often we have guessed it from

$$S(a) = \sum_{n=0}^{\infty} G_a(n, 0) - \sum_{k=0}^{\infty} F_a(0, k)$$

without finding the proof, so our proofs are incomplete. We will explain with more detail identities (1) and (2). For the other identities we will limit ourselves to indicate the WZ-pairs which essentially encapsulate each proof because they are all similar.

**2.1. Identity 1.** Let

$$f(a) = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(a + \frac{1}{2})_n^3}{(a+1)_n^3} [6(n+a) + 1]$$

then we have

$$(1) \quad f(a) = 8a \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(a+1)_n^2}$$

and also

$$(2) \quad f(a) = \frac{4}{\pi} \cdot \frac{4^a}{\cos^2 \pi a} \cdot \frac{1_a^3}{(\frac{1}{2})_a^3} + \frac{16a^2}{2a-1} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (a + \frac{1}{2})_n}{(a+1)_n (\frac{3}{2} - a)_n}.$$

From (1) we easily get the evaluation:

$$f\left(\frac{1}{2}\right) = \frac{\pi^2}{2}$$

and from (2)

$$f(0) = \frac{4}{\pi}, \quad f'(0) = \frac{32}{\pi} \ln 2, \quad f''(0) = \frac{4}{\pi} (64 \ln^2 2 - 3\pi^2).$$

*Proof.* The proof of (1) is essentially encapsulated in the following WZ-pair:

$$\begin{aligned} F(n, k) &= 8 \cdot B(n, k) \cdot n, \\ G(n, k) &= B(n, k) \cdot (6n + 4k + 1), \end{aligned}$$

where

$$B(n, k) = \frac{1}{2^{8n} \cdot 2^{4k}} \cdot \frac{(2k)!^2 \cdot (2n)!^3}{(n+k)!^2 \cdot k!^2 \cdot n!^4}.$$

For proving it we get:

$$\sum_{n=0}^{\infty} G_a(n, 0) = \sum_{n=0}^{\infty} G(n+a, 0) = \frac{(\frac{1}{2})_a^3}{1_a^3} \cdot \frac{1}{4^a} \cdot \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(a + \frac{1}{2})_n^3}{(a+1)_n^3} [6(n+a) + 1],$$

$$\sum_{n=0}^{\infty} F_a(0, k) = \sum_{n=0}^{\infty} F(a, k) = \frac{\left(\frac{1}{2}\right)_a^3}{1_a^3} \cdot \frac{1}{4^a} \cdot \sum_{n=0}^{\infty} 8a \cdot \frac{\left(\frac{1}{2}\right)_n^2}{(a+1)_n^2}.$$

If  $a = 0$  then

$$\lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G(n, k) = \lim_{k \rightarrow \infty} G(0, k) = \frac{4}{\pi}$$

and if  $a > 0$  then

$$\lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G_a(n, k) = 0$$

and the proof of (2) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot \frac{16 \cdot n^2}{2n - 2k - 1},$$

$$G(n, k) = B(n, k) \cdot (6n + 2k + 1),$$

where

$$B(n, k) = \frac{(-1)^k}{2^{8n} \cdot 2^{4k}} \cdot \frac{(2k)! \cdot (2n + 2k)! \cdot \left(n - k - \frac{1}{2}\right)! \cdot (2n)!^2}{k! \cdot (n + k)!^2 \cdot \left(n - \frac{1}{2}\right)! \cdot n!^4}.$$

For proving it we get:

$$\sum_{n=0}^{\infty} G_a(n, 0) = \sum_{n=0}^{\infty} G(n + a, 0) = \frac{\left(\frac{1}{2}\right)_a^3}{1_a^3} \cdot \frac{1}{4^a} \cdot \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(a + \frac{1}{2}\right)_n^3}{(a+1)_n^3} [6(n + a) + 1]$$

$$\sum_{n=0}^{\infty} F_a(0, k) = \sum_{n=0}^{\infty} F(a, k) = \frac{\left(\frac{1}{2}\right)_a^3}{1_a^3} \cdot \frac{1}{4^a} \cdot \frac{16a^2}{2a - 1} \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(a + \frac{1}{2}\right)_k}{(a+1)_k \left(\frac{3}{2} - a\right)_k}$$

and we have guessed that

$$S(a) = \sum_{n=0}^{\infty} G_a(n, 0) - \sum_{k=0}^{\infty} F_a(0, k) = \frac{4}{\pi} \cdot \frac{1}{\cos^2 \pi a}.$$

□

**2.2. Identity 2.** Let

$$f(a) = \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(a + \frac{1}{2}\right)_n^3}{(a+1)_n^3} [42(n + a) + 5],$$

then we have

$$(3) \quad f(a) = 32a \sum_{n=0}^{\infty} \frac{\left(a + \frac{1}{2}\right)_n^2}{(2a+1)_n^2}$$

and also

$$(4) \quad f(a) = \frac{16}{\pi} \cdot \frac{64^a}{\cos^2 \pi a} \cdot \frac{1_a^3}{\left(\frac{1}{2}\right)_a^3} + \frac{128a^2}{2a - 1} \sum_{n=0}^{\infty} \frac{\left(a + \frac{1}{2}\right)_n^2}{(2a+1)_n \left(\frac{3}{2} - a\right)_n}.$$

From (3) we easily get the evaluation:

$$f\left(\frac{1}{2}\right) = \frac{8\pi^2}{3}$$

and from (4)

$$f(0) = \frac{16}{\pi}, \quad f'(0) = \frac{192}{\pi} \ln 2, \quad f''(0) = \frac{16}{\pi} (144 \ln^2 2 - 7\pi^2).$$

*Proof.* The proof of (3) is essentially encapsulated in the following WZ-pair:

$$F(n, k) = 32 \cdot B(n, k) \cdot n, \\ G(n, k) = B(n, k) \cdot \frac{(2n + 2k + 1)^2 \cdot (42n + 4k + 5) - 32 \cdot k \cdot n \cdot (4n + 3k + 2)}{(2n + k + 1)^2},$$

where

$$B(n, k) = \frac{1}{2^{12n} \cdot 2^{4k}} \cdot \frac{(2n + 2k)!^2 \cdot (2n)!^3}{(2n + k)!^2 \cdot (n + k)!^2 \cdot n!^4}$$

and the proof of (4) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot \frac{128 \cdot n^2}{2n - 2k - 1}, \\ G(n, k) = B(n, k) \cdot \frac{(2n + 2k + 1) \cdot (42n + 2k + 5) - 32 \cdot k \cdot n}{2n + k + 1},$$

where

$$B(n, k) = \frac{(-1)^k}{2^{12n} \cdot 2^{4k}} \cdot \frac{(2n + 2k)!^2 \cdot (n - k - \frac{1}{2})! \cdot (2n)!^2}{(n + k)!^2 \cdot (2n + k)! \cdot (n - \frac{1}{2})! \cdot n!^4}.$$

□

**Identity 3.** Let

$$f(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{(a + \frac{1}{2})_n^3}{(a + 1)_n^3} [6(n + a) + 1],$$

then we have

$$(5) \quad f(a) = 4a \sum_{n=0}^{\infty} \frac{(\frac{a}{2} + \frac{1}{4})_n (\frac{a}{2} + \frac{3}{4})_n}{(a + 1)_n^2}$$

and also

$$(6) \quad f(a) = \frac{2\sqrt{2}}{\pi} \cdot \frac{8^a}{\cos \pi a} \cdot \frac{1_a^3}{(\frac{1}{2})_a^3} + \frac{16a^2}{2a - 1} \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{(a + \frac{1}{2})_n^2}{(a + 1)_n (\frac{3}{2} - a)_n}.$$

From (5) we easily get the evaluation:

$$f\left(\frac{1}{2}\right) = 4G$$

and from (6)

$$f(0) = \frac{2\sqrt{2}}{\pi}, \quad f'(0) = \frac{18\sqrt{2}}{\pi} \ln 2, \quad f''(0) = \frac{2\sqrt{2}}{\pi} (81 \ln^2 2 - 4\pi^2).$$

*Proof.* The proof of (5) is essentially encapsulated in the following WZ-pair:

$$F(n, k) = 4 \cdot B(n, k) \cdot n, \\ G(n, k) = B(n, k) \cdot (6n + 4k + 1),$$

where

$$B(n, k) = \frac{(-1)^n}{2^{9n} \cdot 2^{6k}} \cdot \frac{(2n + 4k)! \cdot (2n)!^2}{(n + 2k)! \cdot (n + k)!^2 \cdot n!^3}$$

and the proof of (6) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot \frac{16 \cdot n^2}{2n - 2k - 1},$$

$$G(n, k) = B(n, k) \cdot (6n + 2k + 1),$$

where

$$B(n, k) = \frac{(-1)^k}{2^{9n} \cdot 2^{5k}} \cdot \frac{(2n + 2k)!^2 \cdot (n - k - \frac{1}{2})! \cdot (2n)!}{(n + k)!^3 \cdot (n - \frac{1}{2})! \cdot n!^3}.$$

□

**Identity 4.** Let

$$f(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{(a + \frac{1}{2})_n (a + \frac{1}{4})_n (a + \frac{3}{4})_n}{(a + 1)_n^3} [20(n + a) + 3],$$

then we have

$$(7) \quad f(a) = 16a \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (a + \frac{1}{2})_n}{(a + 1)_n (2a + 1)_n}$$

and also

$$(8) \quad f(a) = \frac{8}{\pi} \cdot \frac{4^a}{\cos \pi a} \cdot \frac{1_a^3}{(\frac{1}{2})_a (\frac{1}{4})_a (\frac{3}{4})_a} + \frac{48a^2}{2a - 1} \sum_{n=0}^{\infty} \frac{1}{4^n} \cdot \frac{(a + \frac{1}{2})_n (2a + \frac{1}{2})_n}{(a + 1)_n (\frac{3}{2} - a)_n}.$$

From (7) we easily get the evaluation:

$$f\left(\frac{1}{2}\right) = 16 \cdot \ln 2$$

and from (8)

$$f(0) = \frac{8}{\pi}, \quad f'(0) = \frac{80}{\pi} \ln 2, \quad f''(0) = \frac{8}{\pi} (100 \ln^2 2 - 5\pi^2).$$

*Proof.* The proof of (7) is essentially encapsulated in the following WZ-pair:

$$F(n, k) = 4 \cdot B(n, k) \cdot n,$$

$$G(n, k) = B(n, k) \cdot \frac{(2n + 2k + 1) \cdot (20n + 4k + 3) - 16 \cdot k \cdot n}{2n + k + 1},$$

where

$$B(n, k) = \frac{(-1)^n}{2^{10n} \cdot 2^{4k}} \cdot \frac{(2k)! \cdot (2n + 2k)! \cdot (4n)!}{(2n + k)! \cdot (n + k)!^2 \cdot k! \cdot n!^2}$$

and the proof of (8) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot \frac{48 \cdot n^2}{2n - 2k - 1},$$

$$G(n, k) = B(n, k) \cdot \frac{(2n + 2k + 1) \cdot (20n + 2k + 3) - 24 \cdot k \cdot n}{2n + 1},$$

where

$$B(n, k) = \frac{(-1)^k}{2^{10n} \cdot 2^{6k}} \cdot \frac{(2n + 2k)! \cdot (4n + 2k)! \cdot (n - k - \frac{1}{2})!}{(2n + k)! \cdot (n + k)!^2 \cdot (n - \frac{1}{2})! \cdot n!^2}.$$

□

**2.3. Identity 5.** Let

$$f(a) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{(a + \frac{1}{2})_n (a + \frac{1}{6})_n (a + \frac{5}{6})_n}{(a+1)_n^3} [154(n+a) + 15],$$

then we have

$$(9) \quad f(a) = 128a \sum_{n=0}^{\infty} \frac{(\frac{a}{2} + \frac{1}{4})_n (\frac{a}{2} + \frac{3}{4})_n}{(a+1)_n (2a+1)_n}$$

and also

$$(10) \quad f(a) = \frac{32\sqrt{2}}{\pi} \cdot \frac{512^a}{27^a \cdot \cos \pi a} \cdot \frac{1_a^3}{(\frac{1}{2})_a (\frac{1}{6})_a (\frac{5}{6})_a} + \frac{512a^2}{2a-1} \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{(a + \frac{1}{2})_n (3a + \frac{1}{2})_n}{(2a+1)_n (\frac{3}{2} - a)_n}.$$

From (9) we easily get the evaluation:

$$f\left(\frac{1}{2}\right) = 128 \ln 2$$

and from (10)

$$f(0) = \frac{32\sqrt{2}}{\pi}, \quad f'(0) = \frac{480\sqrt{2}}{\pi} \ln 2, \quad f''(0) = \frac{32\sqrt{2}}{\pi} (225 \ln^2 2 - 11\pi^2).$$

*Proof.* The proof of (9) is essentially encapsulated in the following WZ-pair:

$$F(n, k) = 128 \cdot B(n, k) \cdot n,$$

$$G(n, k) = B(n, k) \cdot \frac{(2n + 4k + 1) \cdot (154n + 16k + 15) - 384 \cdot k \cdot n}{2n + k + 1},$$

where

$$B(n, k) = \frac{(-1)^n}{2^{15n} \cdot 2^{6k}} \cdot \frac{(2n + 4k)! \cdot (6n)!}{(2n + k)! \cdot (n + 2k)! \cdot (n + k)! \cdot n! \cdot (3n)!}$$

and the proof of (10) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot \frac{512 \cdot n^2}{2n - 2k - 1},$$

$$G(n, k) = B(n, k) \cdot \left[ \frac{(2n + 2k + 1) \cdot (6n + 2k + 3)}{(2n + 1) \cdot (6n + 3k + 3)} \cdot (154n + 6k + 15) - \frac{32kn}{3} \cdot \frac{38n + 14k + 19}{(2n + 1) \cdot (2n + k + 1)} \right],$$

where

$$B(n, k) = \frac{(-1)^k}{2^{15n} \cdot 2^{5k}} \cdot \frac{(2n + 2k)! \cdot (6n + 2k)! \cdot (n - k - \frac{1}{2})!}{(2n + k)! \cdot (3n + k)! \cdot (n + k)! \cdot (n - \frac{1}{2})! \cdot n!^2}.$$

□

**Identity 6.** Let

$$f(a) = \sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{(a + \frac{1}{2})_n (a + \frac{1}{4})_n (a + \frac{3}{4})_n}{(a+1)_n^3} [8(n+a)+1],$$

then we have

$$(11) \quad f(a) = \frac{2\sqrt{3}}{\pi} \cdot \frac{9^a}{\cos 2\pi a} \cdot \frac{1_a^3}{(\frac{1}{2})_a (\frac{1}{4})_a (\frac{3}{4})_a} + \frac{36a^2}{4a-1} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \frac{(\frac{1}{2})_n (a + \frac{1}{2})_n}{(a+1)_n (\frac{3}{2} - 2a)_n}.$$

From (11) we easily get the evaluation:

$$f\left(\frac{1}{2}\right) = \sqrt{3} \cdot \pi$$

and also from (11) we get

$$\begin{aligned} f(0) &= \frac{2\sqrt{3}}{\pi}, & f'(0) &= \frac{4\sqrt{3}}{\pi} (\ln 3 + 4 \ln 2), \\ f''(0) &= \frac{4\sqrt{3}}{\pi} (32 \ln^2 2 + 2 \ln^2 3 + 16 \ln 3 \ln 2 - 3\pi^2). \end{aligned}$$

*Proof.* It is essentially encapsulated in the following WZ-pair:

$$\begin{aligned} F(n, k) &= B(n, k) \cdot \frac{36 \cdot n^2}{4n - 2k - 1}, \\ G(n, k) &= B(n, k) \cdot (8n + 2k + 1), \end{aligned}$$

where

$$B(n, k) = \frac{(-1)^k \cdot 3^k}{2^{8n} \cdot 3^{2n} \cdot 2^{6k}} \cdot \frac{(2k)! \cdot (2n+2k)! \cdot (2n-k-\frac{1}{2})! \cdot (4n)!}{k! \cdot (n+k)!^2 \cdot (2n-\frac{1}{2})! \cdot (2n)! \cdot n!^2}.$$

□

**Identity 7.** Let

$$f(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n} \cdot 3^n} \frac{(a + \frac{1}{2})_n (a + \frac{1}{4})_n (a + \frac{3}{4})_n}{(a+1)_n^3} [28(n+a)+3],$$

then we have

$$(12) \quad \frac{16\sqrt{3}}{3\pi} \cdot \frac{48^a}{\cos \pi a} \cdot \frac{1_a^3}{(\frac{1}{2})_a (\frac{1}{4})_a (\frac{3}{4})_a} + \frac{96a^2}{2a-1} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \frac{(a + \frac{1}{2})_n (2a + \frac{1}{2})_n}{(2a+1)_n (\frac{3}{2} - a)_n}.$$

From (12) we get:

$$\begin{aligned} f(0) &= \frac{16\sqrt{3}}{3\pi}, & f'(0) &= \frac{16\sqrt{3}}{3\pi} (\ln 3 + 12 \ln 2), \\ f''(0) &= \frac{16\sqrt{3}}{3\pi} (144 \ln^2 2 + \ln^2 3 + 24 \ln 3 \ln 2 - 9\pi^2). \end{aligned}$$

*Proof.* It is essentially encapsulated in the following WZ-pair:

$$\begin{aligned} F(n, k) &= B(n, k) \cdot \frac{96 \cdot n^2}{2n - 2k - 1}, \\ G(n, k) &= B(n, k) \cdot \frac{(2n + 2k + 1) \cdot (28n + 2k + 3) - 24 \cdot k \cdot n}{2n + k + 1}, \end{aligned}$$

where

$$B(n, k) = \frac{(-1)^k \cdot 3^k}{2^{12n} \cdot 3^n \cdot 2^{6k}} \cdot \frac{(2n+2k)! \cdot (4n+2k)! \cdot (n-k-\frac{1}{2})! \cdot (2n)!}{(2n+k)!^2 \cdot (n+k)! \cdot (n-\frac{1}{2})! \cdot n!^3}.$$

□

### 3. IDENTITIES FOR A NEW KIND OF EXTENDED SERIES

We consider an extension with a variable  $a$  of some new series for  $1/\pi^2$  obtained by the author [5-7] and get some identities. Once we find an adequate WZ-pair then the difficulty of the proof consist of getting the function

$$S(a) = \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G_a(n, k)$$

but fortunately for the identities in this section we can commute the limit with the sum allowing us to obtain the function  $S(a)$ . We will prove with full detail identity (14). For the other identities we will limit ourselves to indicate the WZ-pairs which essentially encapsulate each proof because they are all similar.

**3.1. Identity 8.** Let  $f(a)$  and  $g(a)$  be the functions:

$$f(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{(a + \frac{1}{2})_n^5}{(a+1)_n^5} [20(n+a)^2 + 8(n+a) + 1],$$

$$g(a) = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(a + \frac{1}{2})_n^3}{(a+1)_n^3} [6(n+a) + 1],$$

then we have

$$(13) \quad f(a) = 8a \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^4}{(a+1)_n^4} (4n+2a+1)$$

and

$$(14) \quad f(a) = \frac{2}{\pi} \cdot \frac{1}{\cos \pi a} \cdot \frac{1_a^2}{(\frac{1}{2})_a^2} \cdot g(a) + \frac{32a^3}{2a-1} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2 (a + \frac{1}{2})_n}{(a+1)_n^2 (\frac{3}{2} - a)_n}$$

and also

$$(15) \quad f(a) = \frac{8}{\pi^2} \cdot \frac{4^a}{\cos^3 \pi a} \frac{1_a^5}{(\frac{1}{2})_a^5} + \frac{32}{\pi} \cdot \frac{1}{\cos \pi a} \cdot \frac{1_a^2}{(\frac{1}{2})_a^2} \cdot \frac{a^2}{2a-1} \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (a + \frac{1}{2})_n}{(a+1)_n (\frac{3}{2} - a)_n} \\ + \frac{32a^3}{2a-1} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2 (a + \frac{1}{2})_n}{(a+1)_n^2 (\frac{3}{2} - a)_n}.$$

From (13) we easily get the evaluation:

$$f\left(\frac{1}{2}\right) = 7\zeta(3).$$

From (14) we get

$$\lim_{a \rightarrow 0} \left[ \frac{f(a)}{a^3} - \frac{2}{\pi} \cdot \frac{1}{\cos \pi a} \cdot \frac{1_a^2}{(\frac{1}{2})_a^2} \cdot \frac{g(a)}{a^3} \right] = -32 \cdot \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n \cdot (2n+1)} = -128 \cdot \frac{G}{\pi}.$$



And from (15) we get

$$f(0) = \frac{8}{\pi^2}, \quad f'(0) = \frac{96}{\pi^2} \ln 2, \quad f''(0) = \frac{64}{3\pi^2} (54 \ln^2 2 - \pi^2).$$

*Proof.* The proof of (13) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot 8n \cdot (2n + 4k + 1),$$

$$G(n, k) = B(n, k) \cdot (20n^2 + 8n + 1 + 24kn + 8k^2 + 4k),$$

where

$$B(n, k) = \frac{(-1)^n}{2^{12n} \cdot 2^{8k}} \cdot \frac{(2k)!^4 \cdot (2n)!^5}{(n+k)!^4 \cdot k!^4 \cdot n!^6}.$$

The proof of (14) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot \frac{32 \cdot n^3}{2n - 2k - 1},$$

$$G(n, k) = B(n, k) \cdot (20n^2 + 12kn + 8n + 2k + 1),$$

where

$$B(n, k) = \frac{(-1)^k \cdot (-1)^n}{2^{12n} \cdot 2^{6k}} \cdot \frac{(2n)!^4 \cdot (2n+2k)! \cdot (2k)!^2 \cdot (n-k-\frac{1}{2})!}{n!^7 \cdot (n+k)!^3 \cdot k!^2 \cdot (n-\frac{1}{2})!}.$$

For proving it we get:

$$\sum_{n=0}^{\infty} G(n+a, 0) = \frac{(\frac{1}{2})_a^5}{1_a^5} \cdot \frac{\cos \pi a}{4^a} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{(a+\frac{1}{2})_n^5}{(a+1)_n^5} [20(n+a)^2 + 8(n+a) + 1],$$

$$\sum_{k=0}^{\infty} F(a, k) = \frac{(\frac{1}{2})_a^5}{1_a^5} \cdot \frac{\cos \pi a}{4^a} \cdot \frac{32a^3}{2a-1} \cdot \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2 (a+\frac{1}{2})_k}{(a+1)_k^2 (\frac{3}{2}-a)_k}.$$

For obtaining the limit:

$$\lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G_a(n, k),$$

we proceed as follows:

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G_a(n, k) &= \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G_a(n, k+1) \\ &= \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} \left[ G_a(n, 0) \prod_{j=0}^k \frac{G_a(n, j+1)}{G_a(n, j)} \right] \\ &= \sum_{n=0}^{\infty} \lim_{k \rightarrow \infty} \left[ G_a(n, 0) \prod_{j=0}^k \frac{G_a(n, j+1)}{G_a(n, j)} \right] \\ &= \frac{2}{\pi} \cdot \frac{(\frac{1}{2})_a^3}{1_a^3} \cdot \frac{1}{4^a} \cdot \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(a+\frac{1}{2})_n^3}{(a+1)_n^3} [6(n+a) + 1]. \end{aligned}$$

For proving (15) just use (14) and (2). □

**3.2. Identity 9.** Let  $f(a)$  and  $g(a)$  be the functions:

$$f(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{\left(a + \frac{1}{2}\right)_n^5}{(a+1)_n^5} [820(n+a)^2 + 180(n+a) + 13],$$

$$g(a) = \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(a + \frac{1}{2}\right)_n^3}{(a+1)_n^3} [42(n+a) + 5],$$

then we have

$$(16) \quad f(a) = 128a \sum_{n=0}^{\infty} \frac{\left(a + \frac{1}{2}\right)_n^4}{(2a+1)_n^4} (4n+6a+1)$$

and

$$(17) \quad f(a) = \frac{8}{\pi} \cdot \frac{16^a}{\cos \pi a} \cdot \frac{1_a^2}{\left(\frac{1}{2}\right)_a^2} \cdot g(a) + \frac{2048a^3}{2a-1} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} + a\right)_n^3}{(2a+1)_n^2 \left(\frac{3}{2} - a\right)_n}$$

and also

$$(18) \quad \begin{aligned} f(a) &= \frac{128}{\pi^2} \cdot \frac{1024^a}{\cos^3 \pi a} \cdot \frac{1_a^5}{\left(\frac{1}{2}\right)_a^5} \\ &+ \frac{1024}{\pi} \cdot \frac{16^a}{\cos \pi a} \cdot \frac{1_a^2}{\left(\frac{1}{2}\right)_a^2} \cdot \frac{a^2}{2a-1} \cdot \sum_{n=0}^{\infty} \frac{\left(a + \frac{1}{2}\right)_n^2}{(2a+1)_n \left(\frac{3}{2} - a\right)_n} \\ &+ \frac{2048a^3}{2a-1} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} + a\right)_n^3}{(2a+1)_n^2 \left(\frac{3}{2} - a\right)_n}. \end{aligned}$$

From (16) we easily get the evaluation [2]:

$$f\left(\frac{1}{2}\right) = 256\zeta(3).$$

From (17) and using [1] the identity:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n \cdot (2n+1)} = \frac{4G}{\pi},$$

we get

$$\lim_{a \rightarrow 0} \left[ \frac{f(a)}{a^3} - \frac{8}{\pi} \cdot \frac{16^a}{\cos \pi a} \cdot \frac{1_a^2}{\left(\frac{1}{2}\right)_a^2} \cdot \frac{g(a)}{a^3} \right] = -2048 \cdot \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n \cdot (2n+1)} = -8192 \cdot \frac{G}{\pi}.$$

And from (18) we get

$$f(0) = \frac{128}{\pi^2}, \quad f'(0) = \frac{2560}{\pi^2} \ln 2, \quad f''(0) = \frac{2560}{3\pi^2} (60 \ln^2 2 - \pi^2).$$

*Proof.* The proof of (15) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot 128 \cdot n \cdot (6n + 4k + 1),$$

$$G(n, k) = B(n, k) \cdot \left[ \frac{(2n+2k+1)^4}{(2n+k+1)^4} \cdot (820n^2 + 180n + 13 + 8k^2 + 20k + 72nk) \right. \\ \left. - (296nk^3 + 1056n^2k^2 + 1280n^3k + 528n^4 + 800n^3 + 1344n^2k \right. \\ \left. + 608nk^2 + 28k^3 + 408n^2 + 384nk + 40k^2 + 72n + 16k + 1) \cdot \frac{32nk}{(2n+k+1)^4} \right],$$

where

$$B(n, k) = \frac{(-1)^n}{2^{20n} \cdot 2^{8k}} \cdot \frac{(2n+2k)!^4 \cdot (2n)!^5}{(2n+k)!^4 \cdot (n+k)!^4 \cdot n!^6}$$

and the proof of (16) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot \frac{2048 \cdot n^3}{2n - 2k - 1},$$

$$G(n, k) = B(n, k) \cdot \left[ 820n^2 + 180n + 13 + \frac{k}{(2n+k+1)^2} (1312n^3 + 1340n^2k + 336nk^2 \right. \\ \left. + 1456n^2 + 828nk + 40k^2 + 472n + 79k + 36) \right],$$

where

$$B(n, k) = \frac{(-1)^k \cdot (-1)^n}{2^{20n} \cdot 2^{6k}} \cdot \frac{(2n)!^4 \cdot (2n+2k)!^3 \cdot (n-k-\frac{1}{2})!}{n!^7 \cdot (n+k)!^3 \cdot (2n+k)!^2 \cdot (n-\frac{1}{2})!}.$$

□

**3.3. Identity 10.** Let  $f(a)$  and  $g(a)$  be the functions:

$$f(a) = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{(a+\frac{1}{2})_n^3 (a+\frac{1}{4})_n (a+\frac{3}{4})_n}{(a+1)_n^5} [120(n+a)^2 + 34(n+a) + 3],$$

$$g(a) = \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{(a+\frac{1}{2})_n^3}{(a+1)_n^3} [42(n+a) + 5],$$

then we have

$$(19) \quad f(a) = 32a \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2 (a+\frac{1}{2})_n^2}{(a+1)_n^2 (2a+1)_n^2} (4n+4a+1)$$

and

$$(20) \quad f(a) = \frac{2}{\pi} \cdot \frac{1}{4^a \cdot \cos 2\pi a} \cdot \frac{1_a^2}{(\frac{1}{4})_a (\frac{3}{4})_a} \cdot g(a) + \frac{512a^3}{4a-1} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(a+1)_n^2 (\frac{3}{2}-2a)_n}$$

and also

$$(21) \quad f(a) = \frac{32}{\pi^2} \cdot \frac{16^a}{\cos^2 \pi a} \cdot \frac{1}{\cos 2\pi a} \cdot \frac{1_a^5}{(\frac{1}{2})_a^3 (\frac{1}{4})_a (\frac{3}{4})_a} + \frac{512a^3}{4a-1} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(a+1)_n^2 (\frac{3}{2}-2a)_n} \\ + \frac{256}{\pi} \cdot \frac{1}{4^a \cdot \cos 2\pi a} \cdot \frac{1_a^2}{(\frac{1}{4})_a (\frac{3}{4})_a} \cdot \frac{a^2}{2a-1} \cdot \sum_{n=0}^{\infty} \frac{(a+\frac{1}{2})_n^2}{(2a+1)_n (\frac{3}{2}-a)_n}.$$

From (19) we easily get the evaluation:

$$f\left(\frac{1}{2}\right) = \frac{16\pi^2}{3}.$$

From (20) we get

$$\lim_{a \rightarrow 0} \left[ \frac{f(a)}{a^3} = \frac{2}{\pi} \cdot \frac{1}{4^a \cdot \cos 2\pi a} \cdot \frac{1_a^2}{\left(\frac{1}{4}\right)_a \left(\frac{3}{4}\right)_a} \cdot \frac{g(a)}{a^3} \right] = -512 \cdot \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{16^n \cdot (2n+1)} = -2048 \cdot \frac{G}{\pi}.$$

And from (21) we get

$$f(0) = \frac{32}{\pi^2}, \quad f'(0) = \frac{512}{\pi^2} \ln 2, \quad f''(0) = \frac{64}{3\pi^2} (384 \ln^2 2 - 7\pi^2).$$

*Proof.* The proof of (17) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot 32n \cdot (4n + 4k + 1),$$

$$G(n, k) = B(n, k) \cdot \left[ \frac{120n^2 + 34n + 3 + 32k^2 + 128kn + 16k}{4n + 4k + 1} + k \cdot \frac{32n^3 + 8n^2k + 16n + 6kn + 40n^2 + k + 2}{(4n + 4k + 1)(2n + k + 1)^2} \right],$$

where

$$B(n, k) = \frac{1}{2^{16n} \cdot 2^{8k}} \cdot \frac{(2k)!^2 \cdot (2n + 2k)!^2 \cdot (4n)! \cdot (2n)!^2}{(2n + k)!^2 \cdot (n + k)!^4 \cdot k!^2 \cdot n!^4}$$

and the proof of (18) is essentially encapsulated in the WZ-pair:

$$F(n, k) = B(n, k) \cdot \frac{512 \cdot n^3}{4n - 2k - 1},$$

$$G(n, k) = B(n, k) \cdot (120n^2 + 84kn + 34n + 10k + 3),$$

where

$$B(n, k) = \frac{(-1)^k}{2^{16n} \cdot 2^{6k}} \cdot \frac{(2n)!^2 \cdot (4n)! \cdot (2k)!^3 \cdot (2n - k - \frac{1}{2})!}{n!^6 \cdot (n + k)!^2 \cdot k!^3 \cdot (2n - \frac{1}{2})!}.$$

□

**Conclusion.** We have found a reduction of ten Ramanujan extended series to simpler hypergeometric series. We believe all the other Ramanujan extended series admit as well a reduction but we have not found the adequate WZ-pairs to get them. It would also be interesting to discover some reduction by using experimental techniques because it could give the clue for finding the corresponding WZ-pairs which would prove the identities.

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